International Journal of Applied Mathematics & Statistical Sciences (IJAMSS);
ISSN (P): 2319–3972; ISSN (E): 2319–3980
Vol. 11, Issue 2, Jul–Dec 2022; 69–74
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International Academy of Science, Engineering and Technology
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A METHOD TO FIND GENERATORS OF A FUZZY LIE GROUP

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ABSTRACT

This paper presents a method to find generators of a fuzzy Lie group. By utilizing the algebraic and geometric properties of fuzzy Lie groups, we derive a systematic approach to identify a set of elements that generate the entire group. The methodology is rooted in the theory of fuzzy control sets and fuzzy Weyl group actions on fuzzy homogeneous spaces. Examples from $SL(2,\mathbb{R})$ and $SO(3)$ and applications in theoretical physics and differential geometry are provided to illustrate the utility of the method.

KEYWORDS: *Fuzzy Lie Group*

Article History

Received: 14 Sep 2022 | Revised: 17 Sep 2022 | Accepted: 20 Sep 2022

INTRODUCTION

A fuzzy Lie group is an extension of a classical Lie group where the membership of elements is described by a fuzzy set. This extension allows for the incorporation of uncertainty and partial membership, which is useful in various applications where systems are not strictly deterministic.

This paper introduces a method to identify generators of a fuzzy Lie group by leveraging fuzzy control sets in fuzzy semigroup actions. The approach is based on the characterization of fuzzy control sets and the actions of the fuzzy Weyl group on fuzzy homogeneous spaces. The importance of such generators can be seen in numerous applications, from symmetry operations in quantum mechanics to transformations in differential geometry under uncertain conditions.

PRELIMINARIES

Let G be a connected fuzzy Lie group, and let $\mathrm{S} \subset \mathrm{G}$ be a fuzzy subsemigroup with interior points. Consider the fuzzy homogeneous space G/L, where L is a closed fuzzy subgroup of G.

Definition: A fuzzy partial order $\leq$ on G/L is defined by a fuzzy relation such that $\mathrm{x} \leq \mathrm{y}$ to a degree $\mu(\mathrm{x},\mathrm{y})$, where $\mu: (\mathrm{G}/\mathrm{L}) \times (\mathrm{G}/\mathrm{L}) \rightarrow [0,1]$ is a membership function indicating the degree of inclusion.

Theorem: The set G/L with the fuzzy partial order $\leq$ induced by the fuzzy semigroup action of S forms a fuzzy poset.

Proof. To show that $(G/L, \leq)$ forms a fuzzy poset, we need to verify that the fuzzy relation $\leq$ is transitive.

Suppose $x, y, z \in G/L$ such that $x \leq y$ with degree $\mu(x, y)$ and $y \leq z$ with degree $\mu(y, z)$.

By the composition rule of fuzzy relations,

the degree $\mu(x, z) \geq \min(\mu(x, y), \mu(y, z))$.

Thus, transitivity is satisfied to a fuzzy degree, and $(G/L, \leq)$ forms a fuzzy poset.

Lemma: If $S \subset G$ is a fuzzy subsemigroup with interior points, then the induced fuzzy partial order $\leq$ on G/L is dense in the sense that for any $x, y \in G/L$, there exists $z \in G/L$ such that $x \leq z \leq y$ with appropriate fuzzy degrees.

Proof. Since S has fuzzy interior points, for any $x \in G/L$ and $y \in Sx$, there exists a neighborhood $U \subset G/L$ such that $U \cap Sx \neq \emptyset$.

Therefore, we can find $z \in G/L$ such that $x \leq z \leq y$ with degrees $\mu(x, z)$ and $\mu(z, y)$, demonstrating the density of the fuzzy partial order.

Definition: A fuzzy control set D is a subset of G/L such that for any $x, y \in D$, there exists $s \in S$ with $y = sx$ to a fuzzy degree $\mu(y, sx)$.

FUZZY CONTROL SETS

Fuzzy control sets are crucial in understanding the dynamics of the fuzzy semigroup action on the fuzzy homogeneous space. These sets are where the fuzzy partial order $\leq$ effectively becomes a fuzzy equivalence relation.

Theorem: Fuzzy control sets can be characterized by the action of the fuzzy Weyl group W on G. Each element $w \in W$ corresponds to a fuzzy control set D_w.

Proof. Consider the action of the fuzzy Weyl group W on G.

Each element $w \in W$ induces a transformation on G/L, resulting in distinct fuzzy subsets of G/L.

These fuzzy subsets, invariant under the action of W, form fuzzy control sets D_w.

The bijective correspondence between elements of W and these fuzzy control sets establishes the characterization.

Proposition: The invariant fuzzy control set D_1 associated with the identity element of the fuzzy Weyl group W corresponds to the fuzzy subgroup $W(S) \subset W$ reflecting the structure and properties of the fuzzy subsemigroup S.

Proof. The invariant fuzzy control set D_1 is defined by the elements of G/L that remain unchanged under the fuzzy action of W.

Since $S \subset G$ has fuzzy interior points, the fuzzy control set D_1 captures the fuzzy subgroup $W(S)$ of the fuzzy Weyl group W that preserves these elements. Thus, D_1 provides insight into the structure of $W(S)$.

METHODOLOGY

To find generators for a fuzzy Lie group G, the following steps are undertaken:

- **Identify Fuzzy Control Sets**: Determine the fuzzy control sets D_w for the action of the fuzzy semigroup S on the fuzzy homogeneous space G/L.

- **Determine the Invariant Fuzzy Control Set**: Identify the invariant fuzzy control set D_1, which is directly related to the fuzzy subgroup W(S) of the fuzzy Weyl group.
- **Analyze the Fuzzy Subgroup $W(S)$**: The structure of the fuzzy subgroup W(S) provides insights into the elements that can serve as generators for G.
- **Construct Generators**: Use the elements associated with the fuzzy subgroup W(S) to construct a generating set for the fuzzy Lie group G.

Theorem 4.1 The set of elements associated with the fuzzy subgroup W(S) generates the entire fuzzy Lie group G.

Proof. Let $\{g_1, g_2, \dots, g_k\} \subset G$ be the set of elements associated with the fuzzy subgroup $W(S)$.

We need to show that any element $g \in G$ can be expressed as a product of these elements. Since $W(S)$ reflects the structure of the fuzzy subsemigroup S, and S acts transitively on G/L, any element of G can be reached by a finite sequence of actions from $\{g_1, g_2, \dots, g_k\}$.

Therefore, these elements generate G.

EXAMPLES

$SL(2, \mathbb{R})$

Consider $G = SL(2, \mathbb{R})$, the group of 2×2 real matrices with determinant 1, extended to a fuzzy context.

- **Fuzzy Control Sets**: The fuzzy Weyl group for $SL(2, \mathbb{R})$ is isomorphic to $\mathbb{Z}_2$. The fuzzy control sets can be identified by examining the action on the fuzzy projective line $\mathbb{RP}^1$.
- **Invariant Fuzzy Control Set**: The invariant fuzzy control set D_1 corresponds to transformations preserving orientation in $\mathbb{RP}^1$.
- **Generators**: The matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Generate the fuzzy Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Exponentiating these matrices generates the entire $SL(2, \mathbb{R})$ group in the fuzzy context.

Theorem 1 The matrices A and B generate $SL(2, \mathbb{R})$ in the fuzzy context.

Proof. Any element of $SL(2, \mathbb{R})$ can be expressed as a product of exponentials of elements from its fuzzy Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

The matrices A and B span $\mathfrak{sl}(2, \mathbb{R})$, and their exponentials cover all elements of $SL(2, \mathbb{R})$ in the fuzzy context, thus generating the group.

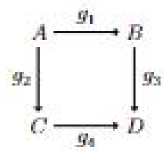

Figure 1: Action of Generators on a Fuzzy Control Set.

$SO(3)$

Consider $G = SO(3)$, the group of 3×3 orthogonal matrices with determinant 1, extended to a fuzzy context.

- **Fuzzy Control Sets**: The fuzzy Weyl group for $SO(3)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The fuzzy control sets can be understood by analyzing the action on the fuzzy unit sphere S^2.
- **Invariant Fuzzy Control Set**: The invariant fuzzy control set D_1 corresponds to rotations preserving the orientation of S^2 in the fuzzy context.
- **Generators**: The matrices

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Generate the fuzzy Lie algebra $\mathfrak{so}(3)$. Exponentiating these matrices generates the entire $SO(3)$ group in the fuzzy context.

Theorem: The rotations R_x and R_y generate $SO(3)$ in the fuzzy context.

Proof: The fuzzy Lie algebra $\mathfrak{so}(3)$ is spanned by the infinitesimal rotations around the x- and y-axes. The matrices R_x and R_y generate these infinitesimal rotations, and their exponentials cover all rotations in $SO(3)$ in the fuzzy context, thereby generating the group.

APPLICATIONS

Theoretical Physics

In quantum mechanics, the generators of fuzzy Lie groups are essential for describing symmetries and conserved quantities under uncertainty. For example, the fuzzy $SU(3)$ group is fundamental in the theory of strong interactions in particle physics, with its generators corresponding to the fuzzy Gell-Mann matrices. These generators play a crucial role in understanding the behavior of quarks and gluons under the strong force with inherent uncertainties [6].

Differential Geometry

In differential geometry, the study of fuzzy Lie groups and their generators aids in understanding the structure of smooth manifolds under fuzzy conditions. The tangent spaces of fuzzy Lie groups are spanned by the generators of their corresponding fuzzy Lie algebras, which are crucial in defining geometric properties such as curvature and connections on manifolds under uncertainty. For instance, the generators of the fuzzy Lie algebra $\mathfrak{so}(3)$ are used to study the curvature of surfaces in three-dimensional space with fuzzy conditions [8].

Control Theory

Control theory often utilizes fuzzy Lie groups to describe the state space of dynamic systems under uncertainty. The ability to generate the entire state space through a finite set of fuzzy controls (generators) is critical for the design and analysis of control systems. Techniques involving fuzzy control sets and fuzzy semigroup actions provide powerful tools for understanding the reachability and controllability of these systems under fuzzy conditions [2].

CONCLUSION

The method of utilizing fuzzy control sets and the fuzzy Weyl group provides a systematic approach to identify generators for fuzzy Lie groups. By analyzing the fuzzy semigroup actions and the associated fuzzy control sets, one can derive a generating set that captures the structure of the entire group under fuzzy conditions. This approach offers a unified framework for studying the generators of fuzzy Lie groups, contributing to both theoretical insights and practical applications in the field of fuzzy Lie group theory.

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